

BINARY QUADRATIC FORMS AND COUNTEREXAMPLES TO HASSE'S LOCAL-GLOBAL PRINCIPLE

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ABSTRACT. After a brief introduction to the classical theory of binary quadratic forms we use these results for proving (most of) the claims made by Pépin in a series of articles on unsolvable quartic diophantine equations, and for constructing families of counterexamples to the Hasse Principle for curves of genus 1 defined by equations of the form $ax^4 + by^4 = z^2$.

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INTRODUCTION

In a series of four articles ([P1, P3, P5, P6]), Théophile Pépin announced the unsolvability of certain diophantine equations of the form $ax^4 + by^4 = z^2$. He did not supply proofs for his claims; a few of his “theorems” were first proved in the author’s article [L1] using techniques that Pépin was not familiar with, such as the arithmetic of ideals¹ in quadratic number fields.

At the time Pépin was studying these diophantine equations, he was working on simplifying Gauss’s theory of composition of quadratic forms (see e.g. [P2, P4]), and it seems natural to look into the theory of binary quadratic forms for approaches to Pépin’s results. In fact we will find that all of Pépin’s claims (and a lot more) can be proved very naturally using quadratic forms.

We start by briefly recalling the relevant facts following Bhargava’s exposition [Bh] of Gauss’s theory (Cox [Co] and Flath [Fl] also provide excellent introductions) for the following reasons:

- It gives us an opportunity to point out some classical references on composition of forms that deserve to be better known; this includes work by Cayley, Riss, Speiser and others.
- Most mathematicians nowadays are unfamiliar with the classical language of binary quadratic forms, and in particular with composition of forms.
- We need some results (such as Thm. 3 below) in a form that is slightly stronger than what can be found in the literature.
- We have to fix the language anyway.

In addition, working with ray class groups instead of forms with nonfundamental discriminants does not save space since we would have to translate the results into the language of forms for comparing them with Pépin’s statements.

¹Ideals were introduced by Dedekind in 1879, but took off only after Hilbert published his Zahlbericht in 1897; its French translation [Hil] started appearing in 1909. Kummer had introduced ideal numbers already in the 1840s, but these were used exclusively for studying higher reciprocity laws and Fermat’s Last Theorem. For investigating diophantine equations, the mathematicians of the late 19th century preferred Gauss’s theory of quadratic forms (see Dirichlet [D1, D2] and Pépin [P2]) to Dedekind’s ideal theory.

Afterwards, we will supply the proofs Pépin must have had in mind. In the final section we will interpret our results in terms of Hasse's Local-Global Principle and the Tate-Shafarevich group of elliptic curves.

1. COMPOSITION OF BINARY QUADRATIC FORMS

A binary quadratic form is a form $Q(x, y) = Ax^2 + Bxy + Cy^2$ in two variables with degree 2 and coefficients $A, B, C \in \mathbb{Z}$; in the following, we will use the notation $Q = (A, B, C)$. The discriminant $\Delta = B^2 - 4AC$ of Q will always be assumed to be a nonsquare. A form (A, B, C) is called primitive if $\gcd(A, B, C) = 1$.

The group $\mathrm{SL}_2(\mathbb{Z})$ of matrices $S = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$ with $r, s, t, u \in \mathbb{Z}$ and determinant $\det S = ru - st = +1$ acts on the set of primitive forms with discriminant Δ via $Q|_S = Q(rx + sy, tx + uy)$; two forms Q and Q' are called equivalent if there is an $S \in \mathrm{SL}_2(\mathbb{Z})$ such that $Q' = Q|_S$. Given $Q = (A, B, C)$, the forms $Q' = Q|_S$ with $S = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ are said to be parallel to Q ; their coefficients are $Q' = (A, B + 2As, C')$ with $C' = Q(s, 1)$. Observe in particular that we can always change B modulo $2A$ (and compute the last coefficient from the discriminant Δ) without leaving the equivalence class of the form. There are finitely many equivalence classes since each form is equivalent to one whose coefficients are bounded by $\sqrt{|\Delta|}$.

A form $Q = (A, B, C)$ represents an integer m primitively if there exist coprime integers a, b such that $m = Q(a, b)$. If Q primitively represents m , then there is an $S \in \mathrm{SL}_2(\mathbb{Z})$ such that $Q|_S = (A', B', C')$ with $A' = m$. In fact, write $Q(r, t) = m$; since $\gcd(r, t) = 1$, there exist $s, u \in \mathbb{Z}$ with $ru - st = 1$; now set $S = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$. This implies that forms representing 1 are equivalent to the principal form Q_0 defined by

$$Q_0(x, y) = \begin{cases} (1, 0, m) & \text{if } \Delta = -4m, \\ (1, 1, m) & \text{if } \Delta = 1 - 4m \end{cases}$$

In fact, forms representing 1 are equivalent to forms $(1, B, C)$, and reducing B modulo 2 shows that they are equivalent to Q_0 .

The set of $\mathrm{SL}_2(\mathbb{Z})$ -equivalence classes of primitive forms (positive definite if $\Delta < 0$) can be given a group structure by introducing composition of forms, which can be most easily explained using Bhargava's cubes². Each cube

$$\mathcal{A} = \begin{array}{ccccc} & & e & \text{---} & f \\ & \swarrow & | & & \swarrow \\ a & \text{---} & & & b \\ & \downarrow & & & \downarrow \\ & & g & \text{---} & h \\ & \swarrow & & & \swarrow \\ c & \text{---} & & & d \end{array}$$

²Historically, Bhargava's cubes occurred in the form of $2 \times 2 \times 2$ -hypermatrices in the work of Cayley (see [C1, C2], or, for a modern account, [GKZ, Chap. 14, Prop. 1.4]), as pairs of bilinear forms as in Eqn. (5) and (6) below (see Gauss [Ga] and Dedekind [De]), as a trilinear form (Dedekind [De] and Weber [We]), and as 2×4 -matrices $\begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix}$ (see Speiser [Sp], Riss [Ri], Shanks [S1, S2, S3], Towber [To], and most other presentations of composition).

of eight integers a, b, c, d, e, f, g, h can be sliced in three different ways (up-down, left-right, front-back):

$$\begin{array}{lll} UD & M_1 = U = \begin{pmatrix} a & e \\ b & f \end{pmatrix}, & N_1 = D = \begin{pmatrix} c & g \\ d & h \end{pmatrix}, \\ LR & M_2 = L = \begin{pmatrix} a & c \\ e & g \end{pmatrix}, & N_2 = R = \begin{pmatrix} b & d \\ f & h \end{pmatrix}, \\ FB & M_3 = F = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, & N_3 = B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}. \end{array}$$

To each slicing we can associate a binary quadratic form $Q_i = Q_i^{\mathcal{A}}$ by putting

$$Q_i(x, y) = -\det(M_i x + N_i y).$$

Explicitly we find

$$\begin{aligned} (1) \quad Q_1(x, y) &= (be - af)x^2 + (bg + de - ah - cf)xy + (dg - ch)y^2, \\ (2) \quad Q_2(x, y) &= (ce - ag)x^2 + (cf + de - ah - bg)xy + (df - bh)y^2, \\ (3) \quad Q_3(x, y) &= (bc - ad)x^2 + (bg + cf - ah - de)xy + (fg - eh)y^2. \end{aligned}$$

These forms all have the same discriminant, and if two of them are primitive (or positive definite), then so is the third.

On the set $\text{Cl}^+(\Delta)$ of equivalence classes of primitive forms with discriminant Δ we can introduce a group structure by demanding that

- The neutral element $[1]$ is the class of the principal form $Q_0(x, y)$.
- We have $([Q_1] \cdot [Q_2]) \cdot [Q_3] = [1]$ if and only if there exists a cube \mathcal{A} with $Q_i = Q_i^{\mathcal{A}}$ for $i = 1, 2, 3$.

Most of the group axioms are easily checked: the cubes

$$\begin{array}{ccc} \begin{array}{c} \begin{array}{cc} 1 & \text{---} & 0 \\ \swarrow & | & \swarrow \\ 0 & \text{---} & 1 \\ | & | & | \\ | & 0 & \text{---} & m \\ \swarrow & & \swarrow \\ 1 & \text{---} & 0 \end{array} \end{array} & \text{or} & \begin{array}{c} \begin{array}{cc} 1 & \text{---} & 1 \\ \swarrow & | & \swarrow \\ 0 & \text{---} & 1 \\ | & | & | \\ | & 1 & \text{---} & \mu \\ \swarrow & & \swarrow \\ 1 & \text{---} & 1 \end{array} \end{array} \end{array}$$

show that $[Q_0][Q_0][Q_0] = [1]$ in the two cases $\Delta = 4m$ and $\Delta = 4m + 1 = 4\mu - 3$, with $\mu = m + 1$.

Next observe that $B \equiv \Delta \pmod{2}$; thus we can put $B = 2b$ if $\Delta = 4m$, and $B = 2b - 1$ if $\Delta = 1 + 4m$. With $b' = 1 - b$ we then find that the two cubes

$$\begin{array}{ccc} \begin{array}{c} \begin{array}{cc} A & \text{---} & -b \\ \swarrow & | & \swarrow \\ 0 & \text{---} & 1 \\ | & | & | \\ | & b & \text{---} & -C \\ \swarrow & & \swarrow \\ 1 & \text{---} & 0 \end{array} \end{array} & \text{and} & \begin{array}{c} \begin{array}{cc} A & \text{---} & b' \\ \swarrow & | & \swarrow \\ 0 & \text{---} & 1 \\ | & | & | \\ | & b & \text{---} & -C \\ \swarrow & & \swarrow \\ 1 & \text{---} & 0 \end{array} \end{array} \end{array}$$

give rise to the quadratic forms $Q_1 = Q_0$, $Q_2 = (A, B, C)$, and $Q_3 = (A, -B, C)$. This shows that the inverse of $[Q]$ for $Q = (A, B, C)$ is the class of $Q^- = (A, -B, C)$. Note in particular that, in general, the classes of Q and Q^- are different (in fact they coincide if and only if their class has order dividing 2), although both Q and

Q^- represent exactly the same integers since $Q(x, y) = Q^-(x, -y)$. Gauss almost apologized for distinguishing the classes of these forms.

The verification of associativity is a little bit involved. Perhaps the simplest approach uses Dirichlet's method of united and concordant forms. Two primitive forms $Q_1 = (A_1, B_1, C_1)$ and $Q_2 = (A_2, B_2, C_2)$ are called concordant if $B_1 = B_2$, $C_1 = A_2C$ and $C_2 = A_1C$ for some integer C . The composition of Q_1 and Q_2 then is the form (A_1A_2, B, C) , as can be seen from the cube

$$\mathcal{A} = \begin{array}{ccccc} & & A_2 & \text{---} & B \\ & \swarrow & | & \searrow & \\ 0 & \text{---} & & A_1 & \\ & \swarrow & | & \searrow & \\ & & 0 & \text{---} & -C \\ & \swarrow & | & \searrow & \\ 1 & \text{---} & & 0 & \end{array}$$

and the associated forms

$$\begin{aligned} Q_1 &= A_1x^2 + Bxy + A_2Cy^2, \\ Q_2 &= A_2x^2 + Bxy + A_1Cy^2, \\ Q_3 &= A_1A_2x^2 - Bxy + Cy^2. \end{aligned}$$

Given three forms, associativity follows immediately if we succeed in replacing the forms by equivalent forms with the same middle coefficients, which is quite easy using the observation that forms represent infinitely many integers coprime to any given number.

Composing two (classes of) forms requires solving³ systems of diophantine equations. All we need in this article is the following observation:

Theorem 1. *The $\text{SL}_2(\mathbb{Z})$ -equivalence classes of primitive forms with discriminant Δ (positive definite forms if $\Delta < 0$) form a group with respect to composition. If $Q_1 = (A_1, B_1, C_1)$ and $Q_2 = (A_2, B_2, C_2)$ are primitive forms with discriminant Δ , and if $e = \gcd(A_1, A_2, \frac{1}{2}(B_1 + B_2))$, then we can always find a form $Q_3 = (A_3, B_3, C_3)$ in the class $[Q_1][Q_2]$ with $A_3 = A_1A_2/e^2$.*

The group of $\text{SL}_2(\mathbb{Z})$ equivalence classes of primitive forms with discriminant Δ is called the class group in the strict sense and is denoted by $\text{Cl}^+(\Delta)$ (the equivalence classes with respect to a suitably defined action by $\text{GL}_2(\mathbb{Z})$ gives rise to the class group $\text{Cl}(\Delta)$ in the wide sense; for negative discriminants, both notions coincide). It is isomorphic to the ideal class group in the strict sense of the order with discriminant Δ inside the quadratic number field $\mathbb{Q}(\sqrt{\Delta})$.

The connection between Bhargava's group law and Gauss composition is provided by the following

Theorem 2. *Let $\mathcal{A} = [a, b, c, d, e, f, g, h]$ be a cube to which three primitive forms $Q_i = Q_i^{\mathcal{A}}$ are attached. Then*

$$(4) \quad Q_1(x_1, y_1)Q_2(x_2, y_2) = Q_3(x_3, -y_3),$$

³For an excellent account of the composition formulas using Dedekind's approach via modules see Lenstra [Le] and Schoof [Sch]. The clearest exposition of the composition algorithm of binary quadratic forms not based on modules is probably Speiser's [Sp]; his techniques also allow to fill the gaps in Shanks' algorithm given in [S3]. Shanks later gave a full version of his composition algorithm which he called NUCOMP.

where x_3 and y_3 are bilinear forms (linear forms in x_1, y_1 and x_2, y_2 , respectively) and are given by

$$(5) \quad x_3 = ex_1x_2 + fx_1y_2 + gx_2y_1 + hy_1y_2,$$

$$(6) \quad y_3 = ax_1x_2 + bx_1y_2 + cx_2y_1 + dy_1y_2.$$

This can be verified e.g. by a computer algebra system; for a conceptual proof, see Dedekind [De] or Speiser [Sp]. The somewhat unnatural minus sign on the right hand side of (4) comes from breaking the symmetry between the forms Q_i ; Dedekind [De] and Weber [We] have shown that

$$Q_1(x_1, y_1)Q_2(x_2, y_2)Q_3(x_3, y_3) = Q_0(x_4, y_4)$$

for certain trilinear forms x_4, y_4 ; this formula preserves the symmetry of the forms involved and makes the group law appear completely natural.

Gauss defined a form Q_3 to be a composite of the forms Q_1 and Q_2 if the identity (4) holds and if (and this additional condition is crucial – it is what allowed Gauss to make form classes into a group⁴) the formulas (1) and (2) hold.

Corollary 1. *Let Δ be a discriminant, r an integer, and p a prime not dividing Δ . Assume that p is primitively represented by a form Q_1 , and that pr is represented primitively by Q_2 . Then we can choose $g \in \{\pm 1\}$ in such a way that p^2r is represented primitively by any form Q_3 with $[Q_1][Q_2]^g = [Q_3]$.*

It is obvious from the Gaussian composition formula (4) that ap^2 is represented by Q_3 and Q'_3 ; what we have to prove is that there exists a *primitive* representation.

As an example illustrating the problem, take $Q_1 = (2, 1, 3)$ and $Q_2 = (2, -1, 3)$. Both forms represent $p = 3$ primitively: we have $3 = Q_1(0, 1) = Q_2(0, 1)$. We also have $[Q_1][Q_2] = [Q_0]$ and $[Q_1][Q_2]^{-1} = [Q_2]$. Both Q_0 and Q_2 represent 9, but $Q_2(2, 1) = 9$ is a primitive representation whereas $Q_0(3, 0) = 9$ is not.

Proof of Cor. 1. We may assume without loss of generality that $Q_1 = (p, B_1, C_1)$ and $Q_2 = (pr, B_2, C_2)$. The composition algorithm shows that $[Q_1][Q_2] = [Q_3]$ for some form $Q_3 = (A_3, B_3, C_3)$ with $A_3 = p^2r/e^2$, where $e = \gcd(p, \frac{1}{2}(B_1 + B_2))$. If $p \nmid \frac{1}{2}(B_1 + B_2)$, then $Q_3(1, 0) = p^2r$ and we are done. If $p \mid \frac{1}{2}(B_1 + B_2)$, replace Q_2 by $Q_2^- = (pr, -B_2, C_2)$; in this case, we find $[Q_1][Q_2^-] = [Q'_3]$ for $Q'_3 = (A_3, B_3, C_3)$ with $A_3 = p^2r/e^2$, where $e = \gcd(p, \frac{1}{2}(B_1 - B_2))$. If $p \nmid \frac{1}{2}(B_1 - B_2)$ we are done; the only remaining problematic case is where p divides both $\frac{1}{2}(B_1 + B_2)$ and $\frac{1}{2}(B_1 - B_2)$, which implies that $p \mid B_1$ and $p \mid B_2$. But then $p \mid (B_1^2 - 4pC_1) = \Delta$, contradicting our assumption. \square

2. GENUS THEORY

Gauss's genus theory characterizes the square classes in $\text{Cl}^+(\Delta)$. Two classes $[Q_1]$ and $[Q_2]$ are said to be in the same genus if there is a class $[Q]$ such that $[Q_1] = [Q_2][Q]^2$. The principal genus is the genus containing the principal class $[Q_0]$; by definition the principal genus consists of all square classes.

⁴Composition of binary quadratic forms can be generalized to arbitrary rings if one is willing to replace quadratic forms by quadratic spaces; see Kneser [Kn] and Koecher [Ko]. Gauss's proof that composition gives a group structure extends without problems to principal ideal domains with characteristic $\neq 2$, and even to slightly more general rings (see e.g. Towber [To]).

Extracting Square Roots in the Class Group. Recall that a discriminant Δ is called fundamental if it is the discriminant of a quadratic number field. Arbitrary discriminants can always be written in the form $\Delta = \Delta_0 f^2$, where Δ_0 is fundamental, and where f is an integer called the conductor of the ring $\mathbb{Z} \oplus \frac{\Delta + \sqrt{\Delta}}{2} \mathbb{Z}$. An elementary technique for detecting squares in the class group $\text{Cl}(\Delta)$ is provided by the following

Theorem 3. *Let Δ_0 be a fundamental discriminant, and assume that Q is a primitive form with discriminant $\Delta = \Delta_0 f^2$. Then the following conditions are equivalent:*

- i) Q represents a square m^2 coprime to f .
- i') Q represents a square m^2 coprime to Δ .
- ii) There exists a primitive form Q_1 with $[Q] = [Q_1]^2$.
- iii) There exist rational numbers x, y with denominator coprime to f such that $Q(x, y) = 1$.

Moreover, if Q represents m^2 primitively, then Q_1 can be chosen in such a way that it represents m primitively.

Proof. Observe that Gauss's equation (4) implies that if Q_1 represents m and Q_2 represents n , then Q_3 represents the product mn . Together with the fact that the primitive form Q_1 represents integers coprime to Δ this shows that ii) implies i).

Let us next show that i) and i') are equivalent. It is clearly sufficient to show that i) implies i'). Assume therefore that $Q(x, y) = A^2$ for coprime integers x, y , and that there is a prime $p \mid \gcd(A, \Delta)$. We claim that $p \mid f$. We know that Q is equivalent to some form (A^2, B, C) , so we may assume that $Q = (A^2, B, C)$.

If p is odd, then $p \mid \Delta = B^2 - 4A^2C$, hence $p \mid B$, $p^2 \mid \Delta$, and finally $p^2 \mid f$ since fundamental discriminants are not divisible by squares of odd primes.

If $p = 2$, then $B = 2b$ and $A = 2a$, and (a^2, b, C) is a form with discriminant $\Delta/4$, showing that $2 \mid f$. Thus i) and i') are equivalent.

For showing that i') \implies ii), assume that Q represents m^2 primitively (cancelling squares shows that Q primitively represents a square), and write $Q = (m^2, B, C)$; Dirichlet composition then shows that $[Q] = [Q_1]^2$ for $Q_1 = (m, B, mC)$; note that if $\Delta < 0$, the form Q_1 is positive definite only for $m > 0$. Since $\gcd(m, \Delta) = \gcd(m, B^2) = 1$, the form Q_1 is primitive.

Finally, i) and iii) are trivially equivalent. \square

We will also need

Corollary 2. *Let Q_1 and Q_2 be forms with discriminant $\Delta = \Delta_0 f^2$. If $Q_j(r_j, s_j) = ax_j^2$ for integers r_j, s_j, x_j ($j = 1, 2$) with $\gcd(x_1 x_2, f) = 1$, then Q_1 and Q_2 belong to the same genus.*

Proof. Any form in the class $[Q_1][Q_2]$ represents a square coprime to f , hence $[Q_1][Q_2] = [Q]^2$. This implies that $[Q_1]$ and $[Q_2]$ are in the same genus. \square

Nonfundamental Discriminants. For negative discriminants, Gauss proved a relation between the class numbers $h(\Delta)$ and $h(\Delta f^2)$. For general discriminants, a similar formula was derived by Dirichlet from his class number formula, and Lipschitz later gave an arithmetic proof of the general result. Since we only consider positive definite forms, we are content with stating a special case of Gauss's result:

Theorem 4. *Let p be a prime, and $\Delta < -4$ a discriminant. Then*

$$(7) \quad h(\Delta p^2) = \left(p - \left(\frac{\Delta}{p}\right)\right) \cdot h(\Delta).$$

The basic tool needed for proving this formula is showing that every form with discriminant Δp^2 is equivalent to a form (A, Bp, Cp^2) , which is “derived” from the form (A, B, C) with discriminant Δ .

Class groups of primitive forms with nonfundamental discriminants occur naturally in the theory of binary quadratic forms, and correspond to certain ray class groups (called ring class groups) in the theory of ideals.

The simplest way of proving (7) is by using the elementary fact that every primitive form with discriminant $\Delta = f^2 \Delta_0$ is equivalent to a form $Q = (A, Bf, Cf^2)$. The form $\overline{Q} = (A, B, C)$ is a primitive form with discriminant Δ_0 from which Q is derived.

3. KAPLANSKY’S “CONJECTURE”

Theorem 3 is related to a question of Kaplansky discussed by Mollin [M1, M2] and Walsh [W1, W2]: Kaplansky claimed that if a prime p can be written in the form $p = a^2 + 4b^2$, then the equation $x^2 - py^2 = a$ is solvable. The assumption $p = a^2 + 4b^2$ implies $p \equiv 1 \pmod{4}$, as well as the solvability of the equation $T^2 - pU^2 = -1$. Since $a^2 = p - 4b^2$, the form $(1, 0, -p)$ with discriminant $\Delta = 4p$ represents a^2 . Since $\gcd(a, 4p) = 1$, there is a form Q with discriminant Δ and $[Q^2] = [Q_0]$ which represents a . Since the class number $h(4p)$ is odd (from (7) we find that $h(4p) = h(p)$ if $p \equiv 1 \pmod{8}$, and $h(4p) = 3h(p)$ if $p \equiv 5 \pmod{8}$; it is a well known result due to Gauss that the class number of forms with prime discriminant is odd), we have $Q \sim Q_0 = (1, 0, -p)$, and the claim follows.

The obvious generalization of Kaplansky’s result is

Proposition 1. *Let m be an integer and p a prime coprime to $2m$. If $p = r^2 + ms^2$, then there is a form Q with the following properties:*

- (1) $\text{disc } Q = 4pm$;
- (2) $Q^2 \sim (p, 0, -m)$;
- (3) Q represents r .

Proof. From $p - ms^2 = r^2$ and $\gcd(r, 2pm) = 1$ we deduce that the form $Q_1 = (p, 0, -m)$ is equivalent to the square of a form representing r . \square

As an example, let $m = 2$ and consider primes $p = r^2 + 2s^2$. Then $p - 2s^2 = r^2$, hence the form $Q = (p, 0, -2)$ with discriminant $\Delta = 8p$ represents the square number r^2 . Thus $Q \sim Q_1^2$ for some form Q_1 representing r .

Now assume that $p \equiv 1 \pmod{8}$. By genus theory, the class of the form Q_1 will be a square if and only if $\left(\frac{2}{r}\right) = \left(\frac{r}{p}\right) = 1$. Since $\left(\frac{2}{r}\right) = \left(\frac{p}{r}\right) = \left(\frac{r}{p}\right)$ by the quadratic reciprocity law, $[Q_1]$ is a square if and only if $\left(\frac{r}{p}\right) = 1$. The congruence $t^2 \equiv -2s^2 \pmod{p}$ implies $\left(\frac{r^2}{p}\right)_4 = \left(\frac{r}{p}\right) = \left(\frac{2}{p}\right)_4 \left(\frac{s}{p}\right)$; writing $s = 2^j u$ for some odd integer u we find $\left(\frac{s}{p}\right) = \left(\frac{2}{p}\right)^j \left(\frac{u}{p}\right) = \left(\frac{p}{u}\right) = 1$. We have proved (see Kaplan [Ka] for proofs of this and a lot of other similar results):

Proposition 2. *Let $p = r^2 + 2s^2 \equiv 1 \pmod{8}$ be a prime. Then the class of the form $Q = (p, 0, -2)$ in $\text{Cl}(8p)$ is a fourth power if and only if $\left(\frac{2}{p}\right)_4 = +1$.*

Note that this does not necessarily imply that the class number $h(8p)$ is divisible by 4 since Q might be equivalent to the principal form. In fact, this always happens if $r = 1$, since then Q represents 1. In this case, we get a unit in $\mathbb{Z}[\sqrt{2p}]$ for free because $p - 2s^2 = 1$ implies that $(\sqrt{p} + s\sqrt{2})^2 = p + 2s^2 + 2s\sqrt{2p}$ is a nontrivial unit. Observe that the field is of Richaud-Degert type since $2p = (2s)^2 + 2$.

4. PÉPIN'S THEOREMS

The simplest among the about 100 theorems stated by Pépin [P1, P3, P5, P6] are the following:

Proposition 3. *In the table below, Q is a positive definite form with discriminant $\Delta = -4m$. For any prime $p \nmid \Delta$ represented by Q (the table below gives two small values of such p), the equations $pX^4 - mY^4 = z^2$ only have the trivial solution.*

Q	(2, 0, 7)	(2, 2, 9)	(4, 0, 5)	(4, 4, 9)	(4, 0, 9)	(3, 0, 13)
m	14	17	20	32	36	39
p	71, 79	13, 89	41, 149	17, 89	13, 73	61, 79

By looking at these results from the theory of binary quadratic forms one is quickly led to observe that the equivalence classes of the forms Q in Pépin's examples are squares but not fourth powers. Such forms occur only for discriminants $\Delta = -4m$ for which $\text{Cl}(\Delta)$ has a cyclic subgroup of order 4. The table below lists all positive $m \leq 238$ with the property that $\text{Cl}(-4m)$ has a cyclic subgroup of order 4, forms Q whose classes are squares but not fourth powers, the structure of the class group, a comment indicating the proof of the result, and a reference to the paper of Pépin's in which it appears.

Pépin must have been aware of the connection between his claims and the structure of the class group for the following reasons:

- (1) The examples he gives in [P3] all satisfy $\text{Cl}(-4m) = [8]$ (the cyclic group of order 8), and those in [P6] satisfy $\text{Cl}(-4m) = [4, 2]$.
- (2) Pépin omits all values of m for which the class number $h(-4m)$ is divisible by 3 or 5, except for three examples given in [P1].
- (3) Most of the misprints in his list concern the middle coefficient of the forms Q , which is sometimes given as half the correct value; a possible explanation is provided by the fact that Gauss used the notation (A, B, C) for the form $Ax^2 + 2Bxy + Cy^2$.

The following result covers all examples in our table:

Theorem 5. *Let m be a positive integer, let Q be a quadratic form with discriminant $\Delta = -4m$, and assume that $[Q]$ is a square, but not a fourth power in $\text{Cl}(\Delta)$. Let $p \nmid \Delta$ be a prime represented by Q . Then the diophantine equation $px^4 - my^2 = z^2$ has nontrivial integral solutions, and if one of the following conditions is satisfied, $px^4 - my^4 = z^2$ has only the trivial solution:*

- (1) Δ is a fundamental discriminant.
- (2) $\Delta = \Delta_0 f^2$ for some fundamental discriminant Δ_0 and an odd squarefree integer f such that $(\frac{\Delta_0}{q}) = -1$ for all $q \mid f$.
- (3) $\Delta = 4\Delta_0$, where $\Delta_0 \equiv 1 \pmod{8}$ is fundamental.
- (4) $\Delta = 4\Delta_0$, where $\Delta_0 = 4n$ is fundamental and $n \equiv 1 \pmod{4}$.
- (5) $\Delta = 16\Delta_0$, where $8 \mid \Delta_0$.

m	Q	$\text{Cl}(-4m)$	comment	ref
14	(2,0,7)	[4]	5.1	[P5]
17	(2,2,9)	[4]	5.1	[P5]
20	(4,0,5)	[4]	5.4	[P5]
32	(4,4,9)	[4]	5.5	[P5]
34	(2,0,17)	[4]	5.1	[P5]
36	(4,0,9)	[4]	5.6	[P1]
39	(3,0,13)	[4]	5.3	[P5]
41	(5,4,9)	[8]	5.1	[P3]
46	(2,0,23)	[4]	5.1	[P5]
49	(2,2,25)	[4]	5.2, f=7	[P5]
52	(4,0,13)	[4]	5.4	[P5]
55	(5,0,11)	[4]	5.3	[P5]
56	(4,4,15)	[4, 2]	$56 = 4 \cdot 14$	[P5]
62	(9,2,7)	[8]	5.1	[P3]
63	(7,0,9)	[4]	5.2, f=3	[P5]
64	(4,4,17)	[4]	5.5	[P5]
65	(10,10,9)	[4, 2]	5.1	[P1]
66	(3,0,22)	[4, 2]	5.1	[P5]
68	(8,2,9) (8,4,9)	[8]	$68 = 4 \cdot 17$	[P3]
69	(6,6,13)	[4, 2]	5.1	[P1]
73	(2,2,37)	[4]	5.1	[P5]
77	(14,14,9)	[4, 2]	5.1	[P5]
80	(9,2,9)	[4, 2]	$80 = 4 \cdot 20$	[P5]
82	(2,0,41)	[4]	5.1	[P5]
84	(4,0,21)	[4, 2]	5.4	[P5]
89	(9, 2, 10), (2,2,45)	[12]	5.1	–
90	(9,0,10)	[4, 2]	5.2, f=3	[P1]
94	(7,4,14)	[8]	5.1	–
95	(9,4,11)	[8]	5.3	–
96	(4,4,25)	[4, 2]	5.5	[P5]
97	(2,2,49)	[4]	5.1	[P5]
98	(9,2,11)	[8]	5.2, f=7	–
100	(4,0,25)	[4]	incorrect	[P5]
111	(7,2,16)	[8]	5.3	[P3]
113	(9,4,13)	[8]	5.1	[P1]
114	(2,0,57) (6,0,19)	[4, 2]	5.1	[P6]
116	(9,2,13), (4,0,29)	[12]	5.4	–
117	(9,0,13)	[4, 2]	5.2, f=3	[P1]
126	(7,0,18)	[4, 2]	$126 = 9 \cdot 14$	[P5]
128	(9,8,16)	[8]	$128 = 4 \cdot 32$	[P1]
132	(4,0,33)	[4, 2]	5.4	[P5]
136	(8,0,17)	[4, 2]	$136 = 4 \cdot 34$	[P5]
137	(9,8,17)	[8]	5.1	[P1]

TABLE 1. Unsolvable Equations $px^4 - my^4 = z^2$.

m	Q	Cl($-4m$)	comment	ref
138	(3,0,46)	[4, 2]	5.1	[P5]
141	(6,6,25)	[4, 2]	5.1	[P1]
142	(2,0,71)	[4]	5.1	[P5]
144	(9, 0, 16)	[4, 2]	$144 = 4 \cdot 36$	–
145	(5,0,29)	[4, 2]	5.1	[P5]
146	(3, 2, 49)	[16]	5.1	–
148	(4,0,37)	[4]	5.4	[P5]
150	(6,0,25)	[4, 2]	incorrect	[P6]
153	(13,4,13) (13,8,13)	[4, 2]	$153 = 9 \cdot 17$	[P6]
154	(11,0,14)	[4, 2]	5.1	[P6]
155	(9, 8, 19), (5,0,31)	[12]	5.4	–
156	(12,0,13)	[4, 2]	$156 = 4 \cdot 39$	[P6]
158	(9,4,18)	[8]	5.1	[P3]
160	(4,4,41)	[4, 2]	5.5	[P6]
161	(9, 2, 18)	[8, 2]	5.1	–
164	(9, 8, 20)	[16]	$164 = 4 \cdot 41$	–
171	(7, 4, 25), (9,0,19)	[12]	5.2, f=3	–
178	(11,6,17)	[8]	5.1	[P3]
180	(9,0,20) (4,0,45)	[4, 2]	$180 = 9 \cdot 20$	[P6]
183	(13,10,16)	[8]	5.3	[P3]
184	(8,8,25)	[4, 2]	$184 = 4 \cdot 46$	[P6]
185	(9,4,21)	[8, 2]	5.1	–
192	(4,2,29) (4,4,49)	[4, 2]	5.5	[P6]
193	(2,2,97)	[4]	5.1	[P5]
194	(11, 4, 18), (6,4,33), (2,0,97)	[20]	5.1	–
196	(8,4,25)	[8]	$196 = 4 \cdot 49$	[P3]
198	(9,0,22)	[4, 2]	5.2, f=3	[P1, P6]
203	(9,4,23), (7,0,29)	[12]	5.3	–
205	(5,0,41)	[4, 2]	5.1	[P6]
206	(9, 2, 23) (14,4,15), 2,0,103)	[20]	5.1	–
208	(16,8,17) (16,16,17)	[4, 2]	$208 = 4 \cdot 52$	[P6]
212	(9, 4,24), (4,0,53)	[12]	5.4	–
213	(6,6,37)	[4, 2]	5.1	[P1]
217	(2, 2, 109)	[4, 2]	5.1	[P1]
219	(12, 6, 19), (3,0,73)	[12]	5.3	–
220	(5, 0, 44)	[4, 2]	$220 = 4 \cdot 55$	[P6]
221	(9,4,25)	[8, 2]	5.1	–
224	(9,2,25)	[8, 2]	5.5	–
225	(9,0,25)	[4, 2]	incorrect	[P6]
226	(11,8,22)	[8]	5.1	–
228	(4,0,57)	[4, 2]	5.4	–
233	(9,2,26), (2,2,117)	[12]	5.1	–
238	(2, 0,119)	[4, 2]	5.1	[P6]

TABLE 2. Unsolvability Equations $px^4 - my^4 = z^2$.

m	Q	Cl($-4m$)	comment	ref
265	(10,10,29)	[4,2]	5.1	[P1]
301	(14,14,25)	[4,2]	5.1	[P1]
360	(9,0,40)	[4,2,2]	$360 = 4 \cdot 90$	[P1]
465	(10,10,49)	[4,2,2]	5.1	[P1]
522	(9,0,58)	[4,2]	5.1, f=3	[P1]
553	(2,2,227)	[4,2]	5.1	[P1]
561	(34,34,25) \sim (25,16,25)	[4,2,2]	5.1	[P1]
609	(42,42,25) \sim (25,8,25)	[4,2,2]	5.1	[P1]
645	(6,6,109)	[4,2,2]	5.1	[P1]
697	(2,2,349)	[4,2]	5.1	[P1]
792	(9,0,88)	[4,2,2]	$792 = 4 \cdot 198$	[P1]
1764	(4,0,441), (25,12,72), (9,0,196)	[8,4]	$1764 = 42^2$	[P1]
3185	(9,2,354)	[16,2,2]	$3185 = 7^2 \cdot 65$	[P1]
4356	(4,0,1089), (148,96,45), (229,74,25)	[12,4]	$4256 = 66^2$	[P1]
4950	(31,28,166), (9,0,550),	[12,2,2]	$4950 = 15^2 \cdot 22$	[P1]
8349	(25,2,254), (49,36,177), 70,62,133)	[24,2,2]	$8349 = 11^2 \cdot 39$	[P1]
256	(16,8,17)	[8]	$256 = 4 \cdot 64$	[P3]
289	(13,12,25)	[8]	incorrect	[P3]
292	(8,4,37)	[8]	$292 = 4 \cdot 73$	[P3]
295	(16,6,19)	[8]	5.3	[P3]
313	(13,10,26)	[8]	5.1	[P3]
252	(9,0,28)	[4,2]	$252 = 4 \cdot 63$	[P6]
282	(3,0,94)	[4,2]	5.1	[P6]
288	(4,2,73) (4,4,73)	[4,2]	$288 = 9 \cdot 32$	[P6]
310	(10,0,31)	[4,2]	5.1	[P6]
322	(2,0,161)	[4,2]	5.1	[P6]
328	(8,0,41)	[4,2]	$328 = 4 \cdot 82$	[P6]
333	(9,0,37)	[4,2]	$333 = 9 \cdot 37$	[P6]
340	(4,0,85)	[4,2]	$340 = 4 \cdot 65$	[P6]
352	(4,2,89) (4,4,89)	[4,2]	5.5	[P6]
372	(4,0,93)	[4,2]	$372 = 4 \cdot 39$	[P6]

TABLE 3. More of Pépin's Examples.

- (6) $\Delta = 4f^2\Delta_0$ for some odd integer f , where $\Delta_0 = 4n$ with $n \equiv 1 \pmod{4}$, and $(\Delta_0/q) = -1$ for all primes $q \mid f$.

Remark 1. In [L1] I have claimed that some proofs can be generalized to show the unsolvability of equations of the form $px^4 - my^2 = z^2$. This is not correct: I have overlooked the possibility that $\gcd(y, z) \neq 1$ in the proofs given there. In fact, consider the equation $71x^4 - 14y^2 = z^2$. We find $71 \cdot 3^2 = 5^4 + 14$, giving rise to a solution $(x, y, z) = (3, 3, 75)$ of $71 \cdot x^4 = z^2 + 14y^2$.

Remark 2. Studying a few of Pépin's examples quickly leads to the conjecture that Theorem 5 holds without any conditions on the discriminant. This is not true: three of Pépin's "theorems" are actually incorrect:

- (1) $m = 100$: here $Q = (4, 0, 25)$ represents the prime $41 = Q(2, 1)$, and the equation $41x^4 - 100y^4 = z^2$ has the solution $(5, 2, 155)$.
- (2) $m = 150$: here $Q = (6, 0, 25)$ represents the prime $31 = Q(1, 1)$, and the equation $31x^4 - 150y^4 = z^2$ has the solution $(5, 3, 85)$.
- (3) $m = 289$: here $Q = (13, 12, 25)$ represents the prime $p = Q(2, -1) = 53$, and the equation $px^4 - 289y^4 = z^2$ has the solution $(x, y, z) = (17, 11, 442)$.

Note that, in these examples, the solutions do not satisfy the condition $\gcd(x, f) = 1$ of Prop. 4 below.

This shows that we have to be careful when trying to generalize Thm. 5 to arbitrary discriminants, and that some sort of condition (like those in (1) – (6)) is necessary.

Remark 3. The obvious generalization of Pépin’s theorems to nonprime values of p does not hold: the form $Q = (2, 0, 7)$ represents $15 = Q(2, 1)$, but the diophantine equation $15x^4 - 14y^4 = z^2$ has the nontrivial (but obvious) solution $(1, 1, 1)$.

As in the proof below, we can deduce that 15^2 is represented by the form Q ; it does not follow, however, that the square roots $(3, \pm 2, 5)$ of $(2, 0, 7)$ represent 15 : in fact, we have $15^2 = Q(9, 3)$, so the representation is imprimitive, and Q is not equivalent to a form with first coefficient 15^2 .

Remark 4. Some of the examples given by Pépin are special cases of others. Consider e.g. the case $m = 80 = 4 \cdot 20$; the derived form of $\overline{Q} = (9, -8, 4) \sim (4, 0, 5)$ with discriminant $-4 \cdot 20$ is the form $Q = (9, -16, 16) \sim (9, 2, 9)$ with discriminant $\Delta = -4m$; its class is easily shown to be a square but not a fourth power, but this is not needed here since every prime represented by Q is also represented by \overline{Q} , which means that the result corresponding to $m = 80$ is a special case of the result for $m = 20$.

The same thing happens for $m = 68, 126, 128, \dots$; the corresponding entries in the tables above are indicated by the comment $m = 4 \cdot n$.

The case $m = -56 = -14 \cdot 4$ is an exception: the derived form of $(7, 0, 2) \sim (2, 0, 7)$ is $Q = (7, 0, 8)$, whose class is a square but not a fourth power in $\text{Cl}^+(\Delta)$. Moreover, the primes that Q represents are also represented by $(2, 0, 7)$.

In addition we have the form $(3, 2, 19) \sim (3, -4, 20)$, and the latter form is derived from $(3, -2, 5)$. This form generates $\text{Cl}(-4 \cdot 14)$, and the square of $(3, -4, 20)$ is equivalent to $(8, 8, 9)$, which underives to $(2, 4, 9) \sim (2, 0, 7)$. The composition of $(8, 8, 9)$ and $(7, 0, 8)$ produces a form equivalent to $(4, 4, 15)$, which is not a square but represents 4.

Observe that the primes p represented by $(4, 4, 15)$ are congruent to 3 mod 4, hence the equations $px^2 - 56y^2 = z^2$ only have solutions with $2 \mid x$ (thus $23x^2 - 56y^2 = z^2$, for example, has the solution $(x, y, z) = (2, 1, 6)$). This implies that the corresponding quartic $23x^4 - 56y^4 = z^2$ does not have a 2-adic solution in a trivial way: $2 \mid x$ implies $4 \mid z$ and $2 \mid y$, so a simple descent shows that this equation does not have a nontrivial solution.

Remark 5. Pépin’s calculations for $m = 114$ are incorrect: here, the square class in $\text{Cl}^+(-4m)$ is generated by $(6, 0, 19)$, whereas Pépin uses $(2, 0, 57)$. Perhaps Pépin went through the forms $(2, 0, n)$ with $n \equiv \pm 1 \pmod{8}$; these forms are contained in square classes if n is prime, or if n has only prime factors $\equiv \pm 1 \pmod{8}$. If the class number $h(-8n)$ is not divisible by 8, the class is not a fourth power.

A direct test whether forms $(2, 0, p)$ for $p \equiv \pm 1 \pmod{8}$ are fourth powers can be performed as follows: write $p = e^2 - 2f^2$; then $Q = (2, 0, p)$ represents $e^2 = 2f^2 + p$, and $[Q]$ is a fourth power if and only if a form with first coefficient $e > 0$ is in the principal genus. If $p \equiv 7 \pmod{8}$, this happens if and only if $(\frac{2}{e}) = +1$, and if $p \equiv 1 \pmod{8}$, if and only if $(\frac{-2}{e}) = +1$. A simple exercise using congruences modulo 16 and quadratic reciprocity shows that $(\frac{2}{e}) = (-1)^{(p+1)/8}$ in the first, and $(\frac{-2}{e}) = (\frac{2}{p})_4$ in the second case.

p	e	f	$(-1)^{(p+1)/8}$	p	e	f	$(2/p)_4$
7	3	1	-1	17	5	2	-1
23	5	1	-1	41	7	2	-1
31	7	3	+1	73	9	2	+1
47	7	1	+1	89	11	4	+1

Thus from Theorem 5 we get the following

Corollary 3. *Let $m = 2q$, where $q \equiv 7 \pmod{16}$ is a prime. For any prime p represented by $(2, 0, q)$, the diophantine equation $px^4 - my^4 = z^2$ has only the trivial solution. The same result holds for primes $q \equiv 1 \pmod{8}$ with $(2/p)_4 = -1$.*

It is an easy exercise to deduce countless other families of similar results from Theorem 5.

The Proof of Theorem 5. The part of Thm. 5 which is easiest to prove generalizes Remark 1:

Lemma 1. *If p is represented by a form Q in the principal genus of $\text{Cl}^+(-4m)$, then the equation $px^4 - my^2 = z^2$ has a nonzero integral solution.*

Proof. The solvability of $px^2 = z^2 + my^2$ follows from genus theory: the prime p is represented by a form in the principal genus, hence any form in the principal genus (in particular the principal form $(1, 0, -m)$) represents p rationally. Multiplying through by x^2 we find $px^4 - m(xy)^2 = (xz)^2$. \square

The main idea behind the proof of Thm. 5 is the content of the following

Proposition 4. *Assume that Q is a form with discriminant $\Delta = -4m$ whose class is a square but not a fourth power in $\text{Cl}^+(\Delta)$. Let $p \nmid \Delta$ be a prime represented by Q , and write $\Delta = \Delta_0 f^2$, where Δ_0 is a fundamental discriminant. Then the diophantine equation $px^4 - my^4 = z^2$ does not have an integral solution (x, y, z) with $\gcd(x, f) = 1$.*

Proof. The form $Q_0 = (1, 0, m)$ represents $px^4 = Q_0(z, y^2)$, and Q represents p . By Cor. 1, the square $p^2 x^4$ is represented primitively by any form in the class $[Q][Q_0]^e = [Q]$ (for some $e = \pm 1$), hence by Q itself. Thus there is a form Q_1 with $Q_1^2 \sim Q$ representing px^2 , which by Cor. 2 is in the same genus as Q . But then Q_1 and Q differ by a square class, which implies that $[Q_1]$ is a square and that $[Q]$ is a fourth power: contradiction. \square

Now we can give the

Proof of Thm. 5. We have to show that if $px^4 - my^4 = z^2$ has a nontrivial integral solution, then there is a solution satisfying $\gcd(x, f) = 1$. Applying Prop. 4 then gives the desired result.

Case (1). If Δ is fundamental, then $f = 1$, and the condition $\gcd(x, f) = 1$ is trivially satisfied.

Case (2). Write $\Delta = 4m$ and $m = q^2n$; then $\Delta_0 = 4n$. Assume that $px^4 - my^4 = z^2$. If $q \mid x$ and $q \nmid y$, then $x = qX$ and $z = qZ$ gives $pq^2X^4 - ny^4 = Z^2$. Reduction modulo q implies $(\frac{-n}{q}) = +1$ contradicting our assumptions since q is odd.

Case (3). If $\Delta = 4\Delta_0$ with $\Delta_0 \equiv 1 \pmod{8}$, then we cannot exclude the possibility that x might be even. Thus in order to guarantee the applicability of Prop. 4 we have to get rid of the factor 4 in $\Delta = -4m$. This is achieved in the following way.

By (7), we have $h(-4m) = (2 - (\frac{-m}{2}))h(-m) = h(m)$ if $-m \equiv 1 \pmod{8}$. Since there is a natural surjective projection $\text{Cl}(-4m) \rightarrow \text{Cl}(-m)$, this implies that $\text{Cl}(-4m) \simeq \text{Cl}(-m)$. But then the form Q projects to a form \overline{Q} with discriminant $-m$ whose class is a square but not a fourth power. Moreover, every integer represented by Q is also represented by \overline{Q} . An application of Prop. 4 with $\text{Cl}^+(\Delta)$ replaced by $\text{Cl}^+(\Delta_0)$ now gives the desired result.

Case (4). Here $m = 4n$ for $n \equiv 1 \pmod{4}$. If $px^4 - 4ny^4 = z^2$ and $x = 2X$, then $z = 2z_1$ and $4pX^4 - ny^4 = z_1^2$. From $0 \equiv z_1^2 + ny^4 \equiv z_1^2 + y^4 \pmod{4}$ we find that $z_1 = 2Z$ and $y = 2Y$ must be even, hence $pX^4 - 4nY^4 = Z^2$. Repeating this if necessary we find a solution with x odd.

Case (5). Here $m = 16n$ with $n \equiv 2 \pmod{4}$. If $px^4 - 16ny^4 = z^2$, then $x = 2X$ and $z = 4Z$, hence $pX^4 - nY^4 = Z^2$.

Since p is represented by a form in the principal genus, we must have $pa^2 = b^2 + mc^2$ for some odd integer a . This implies $p \equiv 1 + mc^2 \equiv 1 \pmod{8}$.

From $1 - 2Y^4 \equiv 1 \pmod{4}$ we deduce that $Y = 2y_1$, hence $pX^4 - 16ny_1^4 = Z^2$. This proves our claim. \square

Case (6). Assume that $m = 4f^2n$ with $n \equiv 1 \pmod{4}$ and $(-n/q) = -1$ for all primes $q \mid f$, and that $px^4 - my^4 = z^2$. If $x = 2X$ and $z = 2z_1$, then $4pX^4 - f^2ny^4 = z_1^2$. If y is odd, then $z_1^2 \equiv 3 \pmod{4}$: contradiction. Thus $y = 2Y$ and $z = 2Z$, hence $pX^4 - 4f^2nY^4 = Z^2$.

If $f = qg$ and $x = qX$, $z = qz_1$, then $pq^2X^4 - 4g^2ny^4 = z_1^2$. Reduction mod q gives $(-n/q) = +1$ contradicting our assumption, except when $q \mid y$. But then $y = qY$ and $z_1 = qZ$, and we find $pX^4 - mY^4 = Z^2$. \square

The abstract argument in case (3) above can be made perfectly explicit:

Lemma 2. Assume that $m \equiv 3 \pmod{4}$, and that $Q = (A, B, C)$ is a form with discriminant $B^2 - 4AC = -4m$. We either have

- (1) $4 \mid B$;
- (2) $B \equiv 2 \pmod{4}$ and $4 \mid C$; or
- (3) $B \equiv 2 \pmod{4}$ and $4 \mid A$.

Then every integer represented by Q is also represented by the form

$$Q' = \begin{cases} (A, A + \frac{B}{2}, \frac{A+B+C}{4}) & \text{in cases (1) and (2)} \\ (\frac{A}{4}, \frac{B}{2}, C) & \text{in case (3)} \end{cases}$$

with discriminant $-m$.

Proof. From $B^2 - 4AC = -4m$ we get $(\frac{B}{2})^2 - AC = -m \equiv 1 \pmod{4}$.

If $4 \mid B$, then $AC \equiv 3 \pmod{4}$, which implies $A + C \equiv 0 \pmod{4}$. Thus in this case, $\frac{B}{2}$ and $\frac{A+B+C}{4}$ are integers.

If $2 \parallel B$, then $1 - AC \equiv 1 \pmod{4}$ implies that $4 \mid AC$. Since the form Q is primitive, we either have $4 \mid A$ or $4 \mid C$.

In cases (1) and (2), substitute $x = X + \frac{1}{2}Y$ and $y = \frac{1}{2}Y$ in $Q(x, y) = Ax^2 + Bxy + Cy^2$ and observe that if x and y are integers, then so are X and Y . We find $Q(x, y) = A(X + \frac{1}{2}Y)^2 + B(X + \frac{1}{2}Y)(\frac{1}{2}Y) + C(\frac{1}{2}Y)^2 = AX^2 + (A + \frac{B}{2})XY + \frac{A+B+C}{4}Y^2$. The coefficients of Q' in these two cases are clearly integers. In case (3), set $x = \frac{1}{2}X$ and $y = Y$. \square

Remark. The proof we have given works with forms in $\text{Cl}(\Delta)^2$; for distinguishing simple squares from fourth powers, one can introduce characters on $\text{Cl}(\Delta)^2/\text{Cl}(\Delta)^4$. This was first done in a special case by Dirichlet, who showed how to express these characters via quartic residue symbols; much later, his results were generalized within the theory of spinor genera by Estes & Pall [EP]. This explains why most proofs of the nonsolvability of equations of the form $ax^4 + by^4 = z^2$ (see in particular [L3]) use quartic residue symbols.

5. EXAMPLES OF LIND-REICHARDT TYPE

The most famous counterexample to the Hasse principle is due to Lind [Li] and Reichardt [Re]; Reichardt showed that the equation $17x^4 - 2y^2 = z^4$ has solutions in every localization of \mathbb{Q} , but no rational solutions except the trivial $(0, 0, 0)$, and Lind constructed many families of similar examples. Below, we will show that our construction also gives some of their examples; we will only discuss the simplest case of fundamental discriminants and are content with the remark that there are similar results in which $-4AC$ is not assumed to be fundamental.

Theorem 6. *Let A and C be coprime positive integers such that $\Delta = -4AC$ is a fundamental discriminant. Then $Q = (A, 0, C)$ is a primitive form with discriminant Δ . If the equivalence class of Q is a square but not a fourth power in $\text{Cl}^+(\Delta)$, then the diophantine equation $Ax^4 - z^4 = Cy^2$ only has the trivial solution $(0, 0, 0)$.*

Proof. We start with the observation that the form $(A, 0, -C)$ represents z^4 . Since $4AC$ is fundamental, the class $[Q]$ must be a fourth power, contradicting our assumptions. \square

The following result is very well known:

Corollary 4. *Let $p \equiv 1 \pmod{8}$ be a prime with $(\frac{2}{p})_4 = -1$. Then the diophantine equation $px^4 - 2y^2 = z^4$ does not have a trivial solution.*

Proof. The form $Q = (p, 0, -2)$ has discriminant $8p$, and Prop. 2 implies that its class is a square but not a fourth power in $\text{Cl}^+(8p)$. \square

6. HASSE'S LOCAL-GLOBAL PRINCIPLE

Some diophantine equations $ax^4 + by^4 = z^2$ can be proved to have only the trivial solution by congruences, that is, by studying solvability in the localizations \mathbb{Q}_p of the rationals. A special case of Hasse's Local-Global Principle asserts that quadratic equations $ax^2 + by^2 = z^2$ have nontrivial solutions in integers (or, equivalently, in rational numbers) if and only if it has solutions in every completion \mathbb{Q}_p .

It is quite easy to see (cf. [AL]) that $ax^4 + by^4 = z^2$ (and therefore also $ax^2 + by^2 = z^2$) has local solutions for every prime $p \nmid 2ab$, so it remains to check solvability in the reals $\mathbb{R} = \mathbb{Q}_\infty$ and in the finitely many \mathbb{Q}_p with $p \mid 2ab$.

Actually we find

Proposition 5. *If $px^2 - my^2 = z^2$ has a rational solution, then the quartic $px^4 - my^4 = z^2$ has a solution in \mathbb{Z}_q for every prime $q > 2$.*

Proof. By classical results (see [AL] for a very simple and elementary proof), $px^4 - my^4 = z^2$ is locally solvable for every prime $q \nmid 2pm$. Thus we only have to look at odd primes $q \mid mp$.

From the solvability of $px^2 - my^2 = z^2$ we deduce that $\left(\frac{p}{q}\right) = +1$ for every odd prime $q \mid m$. But then $\sqrt{p} \in \mathbb{Z}_q$, and we can solve $px^4 - my^4 = z^2$ simply by taking $(x, y, z) = (1, 0, \sqrt{p})$. Moreover, from $\left(\frac{-m}{p}\right) = +1$ we deduce that $\sqrt{-m} \in \mathbb{Z}_p$, hence $(x, y, z) = (0, 1, \sqrt{-m})$ is a p -adic solution of $px^4 - my^4 = z^2$. \square

Thus the quartic $px^4 - my^4 = z^2$, where $p, m > 0$, will be everywhere locally solvable if and only if it is solvable in the 2-adic integers \mathbb{Z}_2 . Now we claim

Proposition 6. *If $m \equiv 2 \pmod{4}$, Pépin's equations are solvable in \mathbb{Z}_2 , hence have solutions in every completion of \mathbb{Q} .*

Proof. Assume first that $m \equiv 2 \pmod{8}$. Then $Q_0 = (1, 0, m)$ (and therefore any form in the principal genus) represents primes $p \equiv 1, 3 \pmod{8}$. If $p \equiv 1 \pmod{8}$, then $p \cdot 1^4 - m \cdot 0^4 \equiv 1 \pmod{8}$ is a square in \mathbb{Z}_2 , and if $p \equiv 3 \pmod{8}$, then $p \cdot 1^4 - m \cdot 1^4 \equiv 3 - 2 \equiv 1 \pmod{8}$ is a square in \mathbb{Z}_2 .

Assume next that $m \equiv 6 \pmod{8}$. Then $Q_0 = (1, 0, m)$ represents only primes $p \equiv \pm 1 \pmod{8}$; the same holds for all forms in the principal genus, in particular the form Q represents only primes $p \equiv \pm 1 \pmod{8}$. For primes $p \equiv 1 \pmod{8}$, the element $p \cdot 1^4 - m \cdot 0^4$ is a square in \mathbb{Z}_2 , so $px^4 - my^4 = z^2$ is solvable in \mathbb{Z}_2 . If $p \equiv 7 \pmod{8}$, then $p \cdot 1^4 - m \cdot 1^4 = p - m \equiv 7 - 6 \equiv 1 \pmod{8}$ is a square in \mathbb{Z}_2 . \square

A similar but simpler proof yields

Proposition 7. *If $m \equiv 0 \pmod{8}$, Pépin's equations are solvable in \mathbb{Z}_2 , hence have solutions in every completion of \mathbb{Q} .*

In fact, primes represented by a form in the principal genus are $\equiv 1 \pmod{8}$, and for these, showing the 2-adic solvability is trivial. Similarly we can show

Proposition 8. *If $m \equiv 1 \pmod{8}$, then $p \equiv 1 \pmod{4}$, and Pépin's equations are solvable in \mathbb{Z}_2 (hence in every completion of \mathbb{Q}) if $p \equiv 1 \pmod{8}$.*

Proof. Since p is represented by some form in the principal genus, we must have $pt^2 = r^2 + ms^2$ for integers r, s, t , with t coprime to $4m$. This implies that t is odd, and that r or s is even, hence we must have $p \equiv pt^2 \equiv 1 \pmod{4}$ as claimed.

We have to study the solvability of $px^4 - my^4 = z^2$ in \mathbb{Z}_2 . If $p \equiv 1 \pmod{8}$, then $\sqrt{p} \in \mathbb{Z}_2$, and we have the 2-adic solution $(x, y, z) = (1, 0, \sqrt{p})$. \square

Thus although not all of Pépin's examples are counterexamples to Hasse's Local-Global Principle, his construction gives infinite families of equations $px^4 - my^4 = z^2$ which have local solutions everywhere but only the trivial solution in integers. As is well known, such equations represent elements of order 2 in the Tate-Shafarevich

group of certain elliptic curves, and it is actually quite easy to use Pépin's construction to find Tate-Shafarevich groups with arbitrarily high 2-rank (see [L2] for the history and a direct elementary proof of this result).

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REFERENCES

- [AL] W. Aitken, F. Lemmermeyer, *Counterexamples to the Hasse Principle: An Elementary Introduction*, Amer. Math. Monthly (2011), to appear
- [Bh] M. Bhargava, *Gauss composition and generalizations*, Algorithmic number theory (Sydney, 2002), 1–8, Lecture Notes in Comput. Sci., 2369, Springer, Berlin, 2002
- [C1] A. Cayley, *On the theory of linear transformations*, Cambridge Math. J. **4** (1845), 1–16; Coll. Math. Papers I (1889), 80–94
- [C2] A. Cayley, *Mémoire sur les hyperdéterminants*, J. Reine angew. Math. **30** (1846), 1–37
- [Co] D. Cox, *Primes of the Form $x^2 + ny^2$* , Wiley 1989
- [De] R. Dedekind, *Über trilineare Formen und die Komposition der binären quadratischen Formen*, J. Reine Angew. Math. **129** (1905), 1–34
- [D1] P.G.L. Dirichlet, *Mémoire sur l'impossibilité de quelques équations indéterminées du cinquième degré*, 1825; Werke I, 1–20
- [D2] P.G.L. Dirichlet, *Mémoire sur l'impossibilité de quelques équations indéterminées du cinquième degré*, J. Reine Angew. Math. **3** (1828), 354–375; Werke I, 21–46
- [EP] D.R. Estes, G. Pall, *Spinor genera of binary quadratic forms*, J. Number Theory **5** (1973), 421–432
- [Fl] D. Flath, *Introduction to number theory*, Wiley & Sons, New York, 1989
- [Ga] C.F. Gauss, *Disquisitiones Arithmeticae*, Leipzig 1801; French transl. by Pouillet Delisle (1807); reprints 1910, 1953; German transl. by H. Maser (1889); English transl. by A.A. Clarke (1965); 2nd rev. ed. Waterhouse et al. (1986); Spanish transl. by H. Barrantes Campos, M. Josephy and Á. Ruiz Zúñiga (1995)
- [GKZ] I.M. Gelfand, M.M. Kapranov, A.V. Zelevinsky, *Discriminants, resultants, and multidimensional determinants*, Birkhäuser Boston, 1994
- [Hil] D. Hilbert, *Théorie des corps de nombres algébriques*, French transl. by Levy and Got, Ann. Fac. Sci. Toulouse **1** (1909), 257–328; **2** (1910), 225–456; **3** (1911), 1–62
- [Ka] P. Kaplan, *Sur le 2-groupe des classes d'idéaux des corps quadratiques*, J. Reine Angew. Math. **283/284** (1976), 313–363
- [Kn] M. Kneser, *Composition of binary quadratic forms*, J. Number Theory **15** (1982), no. 3, 406–413
- [Ko] M. Koecher, *On endomorphisms of degree 2*, Proc. Indian Acad. Sci. **97** (1987), 179–188
- [L1] F. Lemmermeyer, *A note on Pépin's counter examples to the Hasse principle for curves of genus 1*, Abh. Math. Sem. Hamburg **69** (1999), 335–345
- [L2] F. Lemmermeyer, *On Tate-Shafarevich groups of some elliptic curves*, Proc. Conf. Graz 1998, (2000), 277–291
- [L3] F. Lemmermeyer, *Some families of non-congruent numbers*, Acta Arith. **110** (2003), 15–36
- [Le] H.W. Lenstra, *On the calculation of regulators and class numbers of quadratic fields*, Number theory days, 1980 (Exeter, 1980), 123–150, Cambridge 1982
- [Li] C.-E. Lind, *Untersuchungen über die rationalen Punkte der ebenen kubischen Kurven vom Geschlecht Eins*, Diss. Univ. Uppsala 1940
- [M1] R. Mollin, *Proof of some conjectures by Kaplansky*, C.R. Math. Rep. Sci. Canada **23** (2001), 60–64
- [M2] R. Mollin, *On a generalized Kaplansky conjecture*, Int. J. Contemp. Math. Sciences **2** (2007), 411–416
- [P1] Th. Pépin, *Théorèmes d'analyse indéterminée*, C. R. Acad. Sci. Paris **78** (1874), 144–148
- [P2] T. Pépin, *Sur certains nombres complexes compris dans la formule $a + b\sqrt{-c}$* , J. Math. Pures Appl. (3) **I** (1875), 317–372
- [P3] Th. Pépin, *Théorèmes d'analyse indéterminée*, C. R. Acad. Sci. Paris **88** (1879), 1255–1257

- [P4] Th. Pépin, *Composition des formes quadratiques binaires*, Atti Acad. Pont. Nuovi Lincei **33** (1879/80), 6–73
- [P5] Th. Pépin, *Nouveaux théorèmes sur l'équation indéterminée $ax^4 + by^4 = z^2$* , C. R. Acad. Sci. Paris **91** (1880), 100–101
- [P6] Th. Pépin, *Nouveaux théorèmes sur l'équation indéterminée $ax^4 + by^4 = z^2$* , C. R. Acad. Sci. Paris **94** (1882), 122–124
- [Re] H. Reichardt, *Einige im Kleinen überall lösbare, im Großen unlösbare diophantische Gleichungen*, J. Reine Angew. Math. **184** (1942), 12–18
- [Ri] J. Riss, *La composition des formes quadratiques binaires (d'après Gauss)*, Sémin. Théor. Nomb. Bordeaux (1978), exp. 18, 16pp
- [Sch] R. Schoof, *Quadratic fields and factorization*, Computational methods in number theory, Part II, 235–286, Math. Centre Tracts **155**, Amsterdam 1982
- [S1] D. Shanks, *A matrix underlying the composition of quadratic forms and its implications for cubic extensions*, Notices Amer. Math. Soc. **25** (1978), p. A305
- [S2] D. Shanks, *On Gauss and Composition I*, in *Number Theory and Applications* (R. Mollin, ed.), 1989, 163–178
- [S3] D. Shanks, *On Gauss and Composition II*, in *Number Theory and Applications* (R. Mollin, ed.), 1989, 179–204
- [Sp] A. Speiser, *Über die Komposition der binären quadratischen Formen*, Weber-Festschrift (1912), 375–395
- [To] J. Towber, *Composition of oriented binary quadratic form-classes over commutative rings*, Adv. Math. **36** (1980), 1–107
- [W1] G. Walsh, *On a question of Kaplansky*, Amer. Math. Monthly **109** (2002), no. 7, 660–661
- [W2] G. Walsh, *On a question of Kaplansky II*, Albanian J. Math. **2** (2008), 3–6
- [We] H. Weber, *Über die Komposition der quadratischen Formen*, Gött. Nachr. (1907), 86–100